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Tauberian Theorems On A Scale Of Abel-type Summability Methods

Bruce Brigham Watson

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TAUBERIAN THEOREMS ON A SCALE OF ABEL-TYPE SUMMABILITY METHODS

by

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Submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

Faculty of Graduate Studies
The University of Western Ontario
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ABSTRACT

The Abel method A is one of the fundamental methods of summability. It has been generalized to the Abel-type method A_λ by Borwein in his paper which appeared in 1957. The logarithmic method L was investigated by Borwein in a paper which appeared in 1958. Borwein has shown that L includes A_λ for $\lambda > -1$, and A_μ includes A_λ for $\lambda > \mu > -1$, establishing a scale of methods. In the main body of this thesis, two tauberian theorems on this scale are obtained. In the first, if a series is summable A_μ and the A_λ-mean is slowly decreasing, then the series is also summable by the method A_λ. In the second, if a series is summable L, and the A_λ-mean is slowly decreasing (in a different sense), then the series is also summable A_λ. Additional results obtained include: an integral analogue, with simplified hypotheses, of a theorem originally given by Vijayaraghavan; a tauberian theorem in which the condition is that the A_λ-mean be bounded; a direct proof that the method L includes the method A_λ; and a tauberian theorem for abelian summability methods.

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CHAPTER 1

1.1 *INTRODUCTION*

Let $\sum_{n=0}^{\infty} a_n$ be a series of real numbers and let $\{s_n\}$ denote its associated sequence of partial sums. That is,

$$s_n = a_0+a_1+\cdots+a_n \qquad n \geq 0$$

$$s_n = 0 \qquad\qquad n < 0 .$$

The symbol M is used throughout the thesis to denote a positive constant, independent of the variables under consideration, but not necessarily having the same value at each occurrence.

We adopt the following familiar conventions:

$f = O(g)$ means $|f| < Mg$

$f = o(g)$ means $f/g \to 0$

$f \sim g$ means $f/g \to 1.$

The theorems and lemmas in the thesis are numbered chapterwise, and independently. That is, Theorem 3.1 is the first theorem in Chapter Three, however, there may also be a Lemma 3.1. Relations are numbered according to the chapter and section in which they occur. For example, (3.2.4) is the fourth relation in section two of Chapter Three.

1.2 SUMMABILITY METHODS

A *summability method* is a process P which assigns a *sum* s to the series $\sum_{n=0}^{\infty} a_n$. When s is finite, we say that $\sum_{n=0}^{\infty} a_n$ is P-*summable* to the sum s, and write $\sum_{n=0}^{\infty} a_n = s(P)$.

We shall also say that the sequence $\{s_n\}$ is P-*convergent* to the *limit* s, and write

$$s_n \to s(P).$$

A summability method P is said to be *regular* if $s_n \to s(P)$ whenever the sequence $\{s_n\}$ converges to s in the ordinary sense.

Given two summability methods P and Q, P is said to *include* Q if every sequence convergent by the method Q is also convergent by the method P to the same limit. This inclusion relationship will be written

$$Q \subseteq P.$$

If P includes Q, and Q includes P, then we say that P is *equivalent* to Q and write

$$P \simeq Q.$$

1.3 ABEL-TYPE METHODS AND THE LOGARITHMIC METHOD

Let λ be any real number, and $\varepsilon_0^{\lambda} = 1$. Also set

$$\varepsilon_n^{\lambda} = \binom{n+\lambda}{n} = \frac{(\lambda+1)(\lambda+2)\dots(\lambda+n)}{n!} \qquad n = 1,2,\dots$$

$$\sigma_{\lambda}(y) = (1+y)^{-\lambda-1} \sum_{n=0}^{\infty} \varepsilon_n^{\lambda} s_n \left(\frac{y}{1+y}\right)^n \qquad y > 0$$

$$L(y) = \frac{1}{\log(1+y)} \sum_{n=0}^{\infty} \frac{s_n}{n+1} \left(\frac{y}{1+y}\right)^{n+1} \qquad y > 0.$$

The *Abel-type summability method* (A_{λ}), and the *Logarithmic method* (L) are defined as follows: if the series in $\sigma_{\lambda}(y)$ converges for $y > 0$, and $\sigma_{\lambda}(y)$ tends to a finite limit s as $y \to \infty$, then we say that the sequence $\{s_n\}$ is A_{λ}-convergent to s, and write $s_n \to s(A_{\lambda})$; If the series in L(y) converges for $y > 0$ and L(y) tends to a finite limit s as $y \to \infty$, then we say that $\{s_n\}$ is L-convergent to s and write $s_n \to s(L)$.

The Abel-type methods (A_{λ}) were introduced and studied by Borwein [2]. The summability method (A_0) is the ordinary Abel method (A).

Evidently, we have the following:

(i) $s_n \to s(A_{\lambda})$ if and only if the expression

$$(1-x)^{\lambda+1} \sum_{n=0}^{\infty} \varepsilon_n^{\lambda} s_n x^n$$

converges for all x in the open interval (0, 1) and tends to s as $x \to 1^-$.

(ii) $s_n \to s(L)$ if and only if the expression

$$-\frac{1}{\log(1-x)} \sum_{n=0}^{\infty} \frac{s_n}{n+1} x^{n+1}$$

converges for all x in the open interval (0, 1) and tends to s as $x \to 1^-$.

1.4 REGULARITY AND INCLUSION THEOREMS FOR THE (A_λ) AND (L) METHODS

In this section the regularity of the (A_λ) and (L) summability methods is discussed. Also, the scale of Abel-type methods is developed and the natural inclusion between the (A_λ) and (L) summability methods is stated. For the sake of completeness, proofs of some of these known results are given.

THEOREM 1.1

The (A_λ) method is regular for $\lambda > -1$.

THEOREM 1.2

$$A_{\lambda+\varepsilon} \subseteq A_\lambda \quad \text{for } \lambda > -1, \ \varepsilon > 0.$$

THEOREM 1.3

The (L) method is regular.

THEOREM 1.4

$$A_\lambda \subseteq L \quad \text{for } \lambda > -1.$$

Theorems 1.1 and 1.3 are deduced from the following theorem on positive, regular methods of summability in [7].

THEOREM 1.5

Suppose, for $n = 0,1,2, \ldots$

$$c_n(y) \geq 0 \qquad y > 0$$

$$c_n(y) \to 0 \qquad \text{as } y \to \infty$$

and

$$\sum_{n=0}^{\infty} c_n(y) = 1 \qquad y > 0.$$

If, for $y > 0$

$$T(y) = \sum_{n=0}^{\infty} c_n(y)\, s_n,$$

then $T(y) \to s$ *as* $y \to \infty$ *whenever* $s_n \to s$ $(-\infty \leq s \leq \infty)$.

The $T(y)$ of the above theorem becomes $\sigma_\lambda(y)$ and $L(y)$ according as

$$c_n(y) = (1+y)^{-\lambda-1}\, \epsilon_n^\lambda \left(\frac{y}{1+y}\right)^n$$

or

$$c_n(y) = \frac{1}{(n+1)\log(1+y)} \left(\frac{y}{1+y}\right)^{n+1}.$$

The proof of Theorem 1.2 requires some preliminary lemmas.

LEMMA 1.1

If $\lambda > -1$, $\mu > -1$, *and* $|x| < 1$, *then*

(i) $\sum_{n=0}^{\infty} \epsilon_n^\lambda s_n x^n$ *is absolutely convergent if and only if* $\sum_{n=0}^{\infty} \epsilon_n^\mu s_n x^n$ *is absolutely convergent.*

(ii) $\sum_{n=0}^{\infty} \epsilon_n^\lambda s_n x^n$ *is absolutely convergent if and only if* $\sum_{n=0}^{\infty} \frac{s_n}{n+1} x^n$ *is absolutely convergent.*

PROOF

(i) $\epsilon_n^\lambda \sim \frac{n^\lambda}{\Gamma(\lambda+1)}$ as $n \to \infty$.

Therefore

$$\lim_{n\to\infty} (\epsilon_n^\lambda)^{1/n} = 1.$$

Hence the radius of convergence of $\sum_{n=0}^{\infty} \epsilon_n^\lambda s_n x^n$ is the same as that of $\sum_{n=0}^{\infty} s_n x^n$. Therefore (i) follows.

(ii) $\lim_{n\to\infty} \left(\frac{1}{n+1}\right)^{1/n} = 1.$

Hence, as in (i), $\sum_{n=0}^{\infty} \frac{s_n}{n+1} x^n$ has the same radius of convergence as $\sum_{n=0}^{\infty} s_n x^n$ and the lemma follows.

LEMMA 1.2

For $\lambda > -1$, $y > 0$, $\varepsilon > 0$, *and* $n = 0,1,2,\ldots$ *we have*

$$(1.4.1) \quad \frac{\Gamma(\lambda+\varepsilon+1)}{\Gamma(\lambda+1)\Gamma(\varepsilon)} \varepsilon_n^{\lambda+\varepsilon} y^{-\lambda-\varepsilon} \int_0^y (y-t)^{\varepsilon-1} t^{\lambda+n} (1+t)^{-\lambda-\varepsilon-1-n}\, dt$$

$$= \varepsilon_n^{\lambda} (1+y)^{-\lambda-1} \left(\frac{y}{1+y}\right)^n .$$

PROOF

Set $u = t/1+t$, $x = y/1+y$ in the integral in (1.4.1) to obtain

$$\int_0^y (y-t)^{\varepsilon-1}\, t^{\lambda+n}\, (1+t)^{-\lambda-\varepsilon-1-n}\, dt$$

$$= (1-x)^{1-\varepsilon} \int_0^x (x-u)^{\varepsilon-1}\, u^{\lambda+n}\, du$$

$$= x^{\lambda+\varepsilon+n}\, (1-x)^{1-\varepsilon} \int_0^1 t^{\varepsilon-1} (1-t)^{\lambda+n}\, dt$$

$$= \frac{y^{\lambda+\varepsilon+n}\, \Gamma(\varepsilon)\Gamma(\lambda+n+1)}{(1+y)^{\lambda+1+n}\, \Gamma(\lambda+\varepsilon+n+1)} .$$

Therefore

$$\frac{\Gamma(\lambda+\varepsilon+1)}{\Gamma(\lambda+1)\ \Gamma(\varepsilon)}\ \varepsilon_n^{\lambda+\varepsilon}\ y^{-\lambda-\varepsilon}\int_0^y (y-t)^{\varepsilon-1}\ t^{\lambda+n}\ (1+t)^{-\lambda-\varepsilon-1-n}\ dt$$

$$= \varepsilon_n^{\lambda+\varepsilon}\ \frac{\Gamma(\lambda+1+n)}{\Gamma(\lambda+1)}\ \frac{\Gamma(\lambda+\varepsilon+1)}{\Gamma(\lambda+\varepsilon+1+n)}\ (1+y)^{-\lambda-1}\ \left(\frac{y}{1+y}\right)^n$$

$$= \varepsilon_n^{\lambda}\ (1+y)^{-\lambda-1}\ \left(\frac{y}{1+y}\right)^n.$$

This completes the proof of the lemma.

LEMMA 1.3

If $\lambda > -1$, $\varepsilon > 0$, $y > 0$ *and* $\sum_{n=0}^{\infty} \varepsilon_n^{\lambda+\varepsilon}\ s_n\left(\frac{t}{1+t}\right)^n$ *is convergent for all* $t > 0$, *then*

$$(1.4.2)\qquad \sigma_\lambda(y) = \frac{\Gamma(\lambda+\varepsilon+1)}{\Gamma(\varepsilon)\ \Gamma(\lambda+1)}\ \frac{1}{y^{\lambda+\varepsilon}}\int_0^y (y-t)^{\varepsilon-1}\ t^{\lambda}\ \sigma_{\lambda+\varepsilon}(t)\ dt$$

PROOF

By Lemma 1.1, $\sum_{n=0}^{\infty} \varepsilon_n^{\alpha} s_n\left(\frac{t}{1+t}\right)^n$ is absolutely convergent for all $\alpha > -1$, and for $t > 0$. This justifies the interchange of the integration and summation signs in the right-hand side of (1.4.2). The result then follows immediately from Lemma 1.2.

LEMMA 1.4

Suppose that

(i) $K(u,v) \geq 0$ *for* $u > 0$, $v \geq 0$;

(ii) $\int_0^\infty K(u,v)\ dv \to 1$ *as* $u \to \infty$ *and* $\int_0^M K(u,v)\ dv \to 0$ *as* $u \to \infty$ *for each* $M > 0$;

(iii) $f(t)$ *is continuous for* $t > 0$.

Then, if $F(u) = \int_0^\infty K(u,v)\, f(v)\, dv$ *exists in the Cauchy-Lebesgue sense for each* $u > 0$, *we have*

$$\liminf_{v\to\infty} f(v) \le \liminf_{u\to\infty} F(u) \le \limsup_{u\to\infty} F(u) \le \limsup_{v\to\infty} f(v).$$

Lemma 1.4 is the integral analogue of Theorem 1.5 and is proved by an argument of the standard type (cf. [7], proof of Theorem 9).

Theorem 1.2 now follows immediately from Lemmas 1.3 and 1.4.

Theorem 1.2 and Lemmas 1.2 and 1.3 are due to Borwein [2]. Theorem 1.4 was proved by Borwein in [3] and was a corollary of a more general inclusion theorem on methods of summability based on power series. We give a direct proof of Theorem 1.4 in Chapter 4.

1.5 *ADDITIONAL SUMMABILITY METHODS.*
CESARO SUMMABILITY (C,α)

We say that the sequence $\{s_n\}$ is summable to s by the Cesaro method (C,α) for $\alpha > -1$, and write $s_n \to (C,\alpha)$ if

$$s_n^\alpha = \frac{1}{\varepsilon_n^\alpha} \sum_{v=0}^{n} \varepsilon_{n-v}^{\alpha-1} s_v \to s \text{ as } n \to \infty.$$

Set, for $y > 0$

$$\gamma_\alpha(y) = \frac{1}{\Gamma(\alpha+1)} \frac{y^\alpha}{1+y} \sum_{n=0}^{\infty} s_n^\alpha \left(\frac{y}{1+y}\right)^n.$$

THE $A(C,\alpha)$ *METHOD*

For $\alpha > -1$, we say that $s_n \to s\ A(C,\alpha)$ if $\gamma_\alpha(y)$ converges for $y > 0$ and $\lim_{y\to\infty} \Gamma(\alpha+1) y^{-\alpha} \gamma_\alpha(y) = s$.

In connection with the $A(C,\alpha)$ method we state, without proof, the following known results.

LEMMA 1.5

If $y > 0$, $\alpha > -1$, $\delta > 0$ *then*

$$(1.5.1)\quad \gamma_{\alpha+\delta}(y) = \frac{1}{\Gamma(\delta)} \int_0^y (y-t)^{\delta-1}\, \gamma_\alpha(t)\, dt.$$

For $\alpha \geq 0$, this result is due essentially to Kogbetliantz [9] (see also Lord [11], Borwein [4]).

THEOREM 1.6

If $\alpha > -1$, $\delta > 0$ *then* $A(C,\alpha) \subseteq A(C,\alpha+\delta)$.

This result, for $\alpha \geq 0$, is by Lord [11] and for $\alpha > -1$ by Amir [1].

THEOREM 1.7

If $-1 < \alpha < 1$, *then* $A(C,\alpha) \sim A_{-\alpha}$.

This result is due to Borwein [5]. As a corollary we observe that the $A(C,\alpha)$ method is regular, not only for $\alpha \geq 0$, but for $\alpha > -1$.

THE (A,λ) *METHOD*

Let $\{\lambda_n\}$ be a sequence of real numbers satisfying

$$0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \cdots; \qquad \lambda_n \to \infty .$$

The series $\sum_{n=1}^{\infty} a_n$ is said to be summable to s by the abelian summability method (A,λ) if $\sum_{n=1}^{\infty} a_n e^{-y\lambda_n}$ is convergent for $y > 0$ and tends to s as $y \to 0+$. We then write $s_n \to s(A,\lambda)$.

With respect to the (A,λ) method we observe that if $\lambda_n = n$ $(n = 1,2,3,\ldots)$ then $(A,n) \underset{-}{\sim} A_0$, the ordinary Abel method. In addition, it is well-known (see, for example, [7], Theorem 27) that the (A,λ) method is regular.

1.6 *REMARKS CONCERNING THEOREMS 1.2 and 1.4*

In this section we observe that the inclusions established in Theorems 1.2 and 1.4, are *strict* inclusions, in the sense of the following theorems.

THEOREM 1.8

For $\lambda > \mu > -1$, there exists a sequence $\{s_n\}$ which is (A_μ)-convergent but not (A_λ)-convergent.

PROOF

Let $\{s_n\}$ be the sequence such that

$(1-x)^{\lambda+1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n x^n = \sin \frac{1}{1-x}$ for $0 \leq x < 1$. Now, the power series of $(1-x)^{-\lambda-1} \sin \frac{1}{1-x}$ is convergent for $0 < x < 1$, but since $\sigma_\lambda^*(y) = \sin(1+y)$ oscillates as $y \to \infty$, $\{s_n\}$ is not (A_λ)-convergent.

However if $\lambda > \mu > -1$, we have by (1.4.2) that

$$\sigma_\mu(y) = \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\ \Gamma(\lambda-\mu)} \frac{1}{y^\lambda} \int_0^y (y-t)^{\lambda-\mu-1}\ t^\mu \sin(1+t)\ dt$$

$$= \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\ \Gamma(\lambda-\mu)} \int_0^1 (1-u)^{\lambda-\mu-1}\ u^\mu \sin(1+yu)\ du$$

$$\to 0 \text{ as } y \to \infty$$

by the Riemann-Lebesgue theorem. Hence $s_n \to 0(A_\mu)$ and the proof is complete.

This result is due to Borwein [2]. The following result is also due essentially to Borwein [4].

THEOREM 1.9.

For $\lambda > -1$, *there exists a sequence* $\{s_n\}$ *which is* L-*convergent but not* (A_λ)-*convergent.*

PROOF

Let $\{s_n\}$ be the sequence such that

$$\sum_{n=0}^{\infty} s_n x^n = (1-x)^{-1-i} \quad |x| < 1.$$

Then $\gamma_0(y) = (1+y)^i$. Hence, by Lemma 1.5, for $\lambda > 0$, $y > 0$

$$\Gamma(\lambda) y^{-\lambda} \gamma_\lambda(y) = y^{-\lambda} \int_0^y (y-t)^{\lambda-1} (1+t)^i \, dt$$

$$= y^{-\lambda} \int_{-1}^{y} (y-t)^{\lambda-1} (1+t)^i \, dt - y^{-\lambda} \int_{-1}^{0} (y-t)^{\lambda-1} (1+t)^i \, dt$$

$$= \frac{\Gamma(\lambda)\,\Gamma(1+i)}{\Gamma(\lambda+i+1)} y^{-\lambda} (1+y)^{\lambda+i} - y^{-\lambda} \int_{-1}^{0} (y-t)^{\lambda-1} (1+t)^i \, dt.$$

Now, $y^{-\lambda} \int_{-1}^{0} (y-t)^{\lambda-1} (1+t)^i \, dt \to 0$ as $y \to \infty$. Hence $y^{-\lambda} \gamma_\lambda(y)$ does not tend to a limit as $y \to \infty$, and thus $\{s_n\}$ is not $A(C,\lambda)$-convergent for $\lambda > 0$. By Theorem 1.7, $\{s_n\}$ is not summable (A_λ) for $-1 < \lambda < 0$ and hence by Theorem 1.2 $\{s_n\}$ is not summable (A_λ) for any $\lambda > -1$.

However, as $x \to 1$ in $(0,1)$,

$$\sum_{n=0}^{\infty} \frac{s_n}{n+1} x^{n+1} = \int_0^x (1-t)^{-1-i} \, dt = o\{|\log.(1-x)|\}.$$ Hence

$s_n \to 0(L)$. This completes the proof.

CHAPTER 2

A TAUBERIAN THEOREM

2.1 *INTRODUCTION*

In this chapter we introduce the main theorem, to follow in Chapter Three, by giving an elementary proof of the result in a special case.

2.2 *A DEFINITION*

The real-valued function $f(x)$ is said to be *slowly decreasing* if for each $\varepsilon > 0$, there exists a $\delta > 0$ and an $M > 0$ such that $f(y) - f(x) > -\varepsilon$ whenever $y \geq x \geq M$ and $0 \leq \ln \frac{y}{x} < \delta$.

Equivalently, $f(x)$ is slowly decreasing if

$$\liminf \{f(y) - f(x)\} \geq 0$$

whenever $y \geq x \to \infty$ and $(y-x)/x \to 0$.

2.3 *STATEMENT OF THE THEOREM*

THEOREM 2.1

If $\lambda > -1$, $\sum_{n=0}^{\infty} a_n = s(A_\lambda)$, *and* $\sigma_{\lambda+1}(t)$ *is slowly decreasing, then* $\sum_{n=0}^{\infty} a_n = s(A_{\lambda+1})$.

2.4 *SOME LEMMAS*

In this section we assume that $\lambda > -1$, that $f(t)$ is absolutely continuous on $[0,M]$ for each $M \geq 0$, and set

$$(2.4.1) \quad F(y) = \frac{\lambda+1}{y^{\lambda+1}} \int_0^y t^{\lambda} f(t)\,dt \qquad y > 0.$$

LEMMA 2.1

If $F(y) \to 0$ *as* $y \to \infty$, *then* $f(y) \to 0$ *as* $y \to \infty$ *if and only if*

$$\frac{1}{y^{\lambda+1}} \int_0^y t^{\lambda+1} f'(t)\,dt \to 0 \text{ as } y \to \infty.$$

PROOF

An integration by parts yields

$$F(y) = f(y) - \frac{1}{y^{\lambda+1}} \int_0^y t^{\lambda+1} f'(t)\,dt$$

Hence the result.

LEMMA 2.2

If $F(y) \to s$ *as* $y \to \infty$, *then* $\int_0^{\infty} \frac{dy}{y^{\lambda+2}} \int_0^y t^{\lambda+1} f'(t)\,dt$

exists in the Cauchy-Lebesgue sense.

PROOF

We have

$$\int_0^R \frac{dy}{y^{\lambda+2}} \int_0^y t^{\lambda+1} f'(t)\,dt = \int_0^R t^{\lambda+1} f'(t)\,dt \int_t^R \frac{dy}{y^{\lambda+2}}$$

$$= -\frac{1}{(\lambda+1)\,R^{\lambda+1}} \int_0^R t^{\lambda+1} f'(t)\,dt + \frac{1}{\lambda+1} \int_0^R f'(t)\,dt$$

$$= \frac{1}{R^{\lambda+1}} \int_0^R t^{\lambda} f(t)\,dt - \frac{f(R)}{\lambda+1} + \frac{f(R)}{\lambda+1} - \frac{f(0)}{\lambda+1}$$

$$\to \frac{s - f(0)}{\lambda+1} \text{ as } R \to \infty.$$

The absolute continuity of $f(t)$ allows the inversion of the order of integration and the integration by parts performed here. The result follows.

LEMMA 2.3

If $f(t)$ *is slowly decreasing and* $F(y) \to 0$ *as* $y \to \infty$ *then* $f(y) \to 0$ *as* $y \to \infty$.

PROOF

By Lemma 2.1, it suffices to show

$$(2.4.2) \quad \frac{1}{y^{\lambda+1}} \int_0^y t^{\lambda+1} f'(t)\, dt \to 0 \text{ as } y \to \infty.$$

Suppose (2.4.2) does not hold. Then there exists a sequence $\{R_n\}$ tending to infinity and a $C > 0$ such that either

$$\text{(i)} \quad \int_0^{R_n} t^{\lambda+1} f'(t)\, dt > C R_n^{\lambda+1} \qquad n = 0,1,2,\ldots$$

or

$$\text{(ii)} \quad \int_0^{R_n} t^{\lambda+1} f'(t)\, dt < -C R_n^{\lambda+1} \qquad n = 0,1,2,\ldots$$

Suppose (i) holds.

Since $f(t)$ is slowly decreasing, for a sufficiently small, but fixed, positive ε, say $\frac{C}{4} > \varepsilon > 0$, there exists an $M > 0$ and a $\gamma > 1$ such that $f(t) - f(x) > -\varepsilon$ whenever $x \geq M$ and $x \leq t \leq \gamma x$. Furthermore we may suppose γ is small enough so that $\gamma^{\lambda+1}\varepsilon < \frac{C}{2}$.

Let v satisfy the inequality $M \leq R_n \leq v \leq \gamma R_n$.

By the second mean value theorem for integrals, there exists a u such that $R_n \leq u \leq v \leq \gamma R_n$ and

$$\int_{R_n}^{v} t^{\lambda+1} f'(t)\, dt = v^{\lambda+1} \{f(v) - f(u)\}.$$

Hence

$$\int_{R_n}^{v} t^{\lambda+1} f'(t)\, dt > -\varepsilon\, \gamma^{\lambda+1} R_n^{\lambda+1}$$

$$> -\frac{C}{2} R_n^{\lambda+1}$$

Therefore, when $\gamma R_n \geq y \geq R_n \geq M$

$$\int_0^y t^{\lambda+1} f'(t)\, dt$$

$$= \int_0^{R_n} t^{\lambda+1} f'(t)\, dt + \int_{R_n}^{y} t^{\lambda+1} f'(t)\, dt$$

$$> \frac{C}{2} R_n^{\lambda+1}.$$

Hence, for $R_n \geq M$

$$\int_{R_n}^{\gamma R_n} \frac{dy}{y^{\lambda+2}} \int_0^y t^{\lambda+1} f'(t)\, dt$$

$$> \frac{C}{2} R_n^{\lambda+1} \int_{R_n}^{\gamma R_n} \frac{dy}{y^{\lambda+2}}$$

$$= \frac{C}{2(\lambda+1)} \{1 - \frac{1}{\gamma^{\lambda+1}}\} .$$

Therefore, by Lemma 2.2, $F(y)$ does not tend to a limit, which is a contradiction.

Suppose (ii) holds.

For $\frac{C}{2} > \varepsilon > 0$, there exists an $M > 0$ and a δ satisfying $0 < \delta < 1$ such that $f(x) - f(t) > -\varepsilon$ whenever $x \geq M/\delta$ and $\delta x \leq t \leq x$.

Now, for any u satisfying $M \le \delta R_n \le u \le R_n$, there exists a v satisfying $\delta R_n \le u \le v \le R_n$ such that

$$\int_u^{R_n} t^{\lambda+1} f'(t)\, dt = R_n^{\lambda+1} \{f(R_n) - f(v)\}$$
$$> -\frac{C}{2} R_n^{\lambda+1}.$$

Hence, for $M \le \delta R_n \le y \le R_n$

$$\int_0^y t^{\lambda+1} f'(t)\, dt$$

$$= \int_0^{R_n} t^{\lambda+1} f'(t)\, dt - \int_y^{R_n} t^{\lambda+1} f'(t)\, dt$$

$$< -\frac{C}{2} R_n^{\lambda+1}.$$

Therefore, when $M \le \delta R_n$

$$\int_{\delta R_n}^{R_n} \frac{dy}{y^{\lambda+2}} \int_0^y t^{\lambda+1} f'(t)\, dt$$

$$< -\frac{C}{2} R_n^{\lambda+1} \int_{\delta R_n}^{R_n} \frac{dy}{y^{\lambda+2}}$$

$$= -\frac{C}{2(\lambda+1)} \{\frac{1}{\delta^{\lambda+1}} - 1\}.$$

Again, by Lemma 2.2, F(y) does not tend to a limit as $y \to \infty$, which is a contradiction. This establishes the result.

2.5 *PROOF OF THEOREM 2.1*

It suffices to suppose that $s_n \to O(A_\lambda)$. By (1.4.2) we have

$$\sigma_\lambda(y) = \frac{\lambda+1}{y^{\lambda+1}} \int_0^y t^\lambda \sigma_{\lambda+1}(t)\,dt \qquad y > 0.$$

By hypothesis, $\sigma_{\lambda+1}(t)$ is slowly decreasing, and since $\sigma_{\lambda\pm1}(t)$ is absolutely continuous in $[0,X]$ for each $X \geq 0$, the result follows immediately from Lemma 2.3.

2.6 *COMMENT*

Our object is to establish a more general version of Theorem 2.1 in which the occurrences of $(\lambda+1)$ in Theorem 2.1 are replaced by $(\lambda+\varepsilon)$ for an arbitrary, but fixed, $\varepsilon > 0$. However, we adopt a different technique of proof, since any generalization of the method of this chapter seems to be non-trivial. More specifically, the integral transformation which would replace the $F(y)$ of this chapter adds extra difficulties.

CHAPTER 3

A TAUBERIAN THEOREM OF SLOWLY DECREASING TYPE - I

In this chapter we prove a tauberian theorem on the (A_λ) scale, more general than Theorem 2.1. As a step in that proof we also state and prove an integral analogue, with simplified hypotheses, of a theorem initially proved by Vijayaraghavan [15] and also given by Hardy in [7]. Finally, as a corollary of the main result, we give a tauberian theorem of O-type.

3.1 *STATEMENT OF THE THEOREM*

The theorem we prove is the following.

THEOREM 3.1

For $\lambda > -1$ *and* $\varepsilon > 0$, *if* $\sum_{n=0}^{\infty} a_n = s(A_\lambda)$ *and* $\sigma_{\lambda+\varepsilon}(t)$ *is slowly decreasing, then* $\sum_{n=0}^{\infty} a_n = s(A_{\lambda+\varepsilon})$.

The proof is divided into two stages. In the first stage we show that (A_λ) summability, and the fact that $\sigma_{\lambda+\varepsilon}(t)$ is slowly decreasing imply that $\sigma_{\lambda+\varepsilon}(t)$ is bounded. This will follow by the Vijayaraghavan-type theorem in the next section. In the second stage we go from the boundedness of $\sigma_{\lambda+\varepsilon}(t)$ to $(A_{\lambda+\varepsilon})$ summability, by means of a Wiener tauberian theorem.

3.2 *A VIJAYARAGHAVAN-TYPE RESULT*

In the next three sections we assume the following initial hypotheses.

(i) $K(u,v)$ is defined, real-valued, and non-negative for $u > 0$, $v \geq 0$. Moreover, $\int_0^\infty K(u,v)dv$ exists in the sense of Lebesgue for each $u > 0$.

(ii) $\int_0^\infty K(u,v)dv \to 1$ as $u \to \infty$.

(iii) $f(v)$ is real-valued and continuous for $v \geq 0$.

(iv) $F(u) = \int_0^\infty K(u,v)f(v)dv$ exists in the Cauchy-Lebesgue sense for each $u > 0$.

THEOREM 3.2

Suppose the following conditions hold:

(3.2.1) $\Phi(x)$ *is a real-valued, non-negative, increasing, continuous function defined on* $[0,\infty)$ *such that* $\Phi(x) \to \infty$ *as* $x \to \infty$;

(3.2.2) $\liminf \{f(y) - f(x)\} \geq -\mu > -\infty$ *for some fixed, positive* μ, *whenever* $y > x \to \infty$, *and* $\Phi(y) - \Phi(x) \to 0$;

(3.2.3) $\Phi(x) - \Phi(x-1) \to 0$ *as* $x \to \infty$;

(3.2.4) $\int_0^x K(u,v)dv \to 0$ *whenever* $u > x \to \infty$ *and* $\Phi(u) - \Phi(x) \to \infty$;

(3.2.5) $\int_x^\infty K(u,v)\ \{\Phi(v) - \Phi(x)\}dv \to 0$ *whenever* $x > u \to \infty$ *and* $\Phi(x) - \Phi(u) \to \infty$; *and*

(3.2.6) $F(u) = O(1)$ *for* $u > 0$. *Then* $f(v) = O(1)$ *for* $v > 0$.

3.3 SOME PRELIMINARY LEMMAS

LEMMA 3.1

If (3.2.1) *and* (3.2.4) *hold, then* $f(v) \to s$ *as* $v \to \infty$ *implies* $F(u) \to s$ *as* $u \to \infty$ *for finite or infinite* s.

PROOF

By Lemma 1.4 it suffices to show that for every fixed $M > 0$

$$\int_0^M K(u,v)dv \to 0 \text{ as } u \to \infty.$$

Take $\varepsilon > 0$. By (3.2.4) there exist an $X_1 \geq M > 0$ and an $R_1 > 0$ such that

$$\int_0^{X_1} K(u,v)dv < \varepsilon \text{ whenever}$$

$u > X_1$ and $\phi(u) - \phi(X_1) \geq R_1$. Define U_1 to be that value of u such that $\phi(U_1) = R_1 + \phi(X_1)$. Then for $u \geq U_1$

$$\int_0^M K(u,v)dv \leq \int_0^{X_1} K(u,v)dv < \varepsilon.$$

This completes the proof.

LEMMA 3.2

If (3.2.1) *and* (3.2.2) *hold, then there exist positive constants* M_1, M_2 *such that*

$f(y) - f(x) > -M_1\{\phi(y) - \phi(x)\} - M_2$ *for* $y \geq x \geq 0$.

PROOF

Take $\varepsilon_0 > 0$. By (3.2.2) there exist $X_1 > 0$ and

$\delta > 0$ such that

(3.3.1) $f(y) - f(x) > -\mu - \varepsilon$ whenever $y > x \geq X_1$ and $\Phi(y) - \Phi(x) \leq \delta$.

But there exists an $M_2 > 0$ such that $f(y) - f(x) > -M_2$ whenever $y > x \geq 0$ and $\Phi(y) - \Phi(x) \leq \delta$. For if

(i) $X_1 \geq y > x \geq 0$ then y is clearly bounded above.

If (ii) $y > X_1 \geq x \geq 0$ and $\Phi(y) - \Phi(x) \leq \delta$ then $\Phi(y) - \Phi(X_1) \leq \Phi(y) - \Phi(x) \leq \delta$ and since Φ is increasing to infinity, y is bounded above. In either case, since $f(x)$ is continuous, there exists a positive N_1 such that $f(y) - f(x) > -N_1$. Combining this with (3.3.1) we take $M_2 = \mathrm{MAX}\{\mu+\varepsilon, N_1\}$ to get $f(y) - f(x) > -M_2$ whenever $y > x \geq 0$ and $\Phi(y) - \Phi(x) \leq \delta$. Now fix x,y such that $y > x \geq 0$. Define a sequence $\{x_r\}$ for $r = 0,1,2,\ldots$ such that $x_0 = x$ and x_r satisfies $\Phi(x_r) = \Phi(x_{r-1}) + \delta \quad r=1,2,\ldots$. Since $\Phi(x_r) = \Phi(x_0) + r\delta$ we have $x_r \to \infty$ as $r \to \infty$. Hence, there exists an integer m such that $x_m \leq y < x_{m+1}$. Therefore

$$(3.3.2) \quad f(y) - f(x) = \sum_{r=0}^{m-1} \{f(x_{r+1}) - f(x_r)\} + f(y) - f(x_m).$$

$$> -m\, M_2 - M_2.$$

However, since $m\delta = \Phi(x_m) - \Phi(x_0) \leq \Phi(y) - \Phi(x)$ we have

$$-m \geq -\frac{\Phi(y) - \Phi(x)}{\delta}$$

Therefore by (3.3.2)

$$f(y) - f(x) > -\frac{M_2}{\delta}\{\Phi(y) - \Phi(x)\} - M_2.$$

The desired result follows.

LEMMA 3.3

If (3.2.1) *and* (3.2.5) *hold, then*

$$\int_x^\infty K(u,v)dv \to 0$$

whenever $x > u \to \infty$ *and* $\Phi(x) - \Phi(u) \to \infty$.

PROOF

Take $\varepsilon > 0$. By (3.2.5), there exist an $X_0 > 0$ and $R_0 > 1$ such that

$$\int_x^\infty K(u,v)\ \{\Phi(v) - \Phi(x)\}dv < \varepsilon$$

whenever $x > u > X_0$ and $\Phi(x) - \Phi(u) \geq R_0$. Now take fixed x and u such that

$$x > u > X_0 \text{ and } \Phi(x) - \Phi(u) \geq R_0 + 1.$$

Since $\Phi(x)$ is continuous and increasing, there exists a w satisfying $u < w < x$ and $\Phi(x) - \Phi(w) = 1$. Now

$\Phi(w) - \Phi(u) = \Phi(w) - \Phi(x) + \Phi(x) - \Phi(u) \geq -1 + R_0 + 1 = R_0$.

Hence

$$\begin{aligned}\int_x^\infty K(u,v)dv &= \int_x^\infty K(u,v)\ \{\Phi(x) - \Phi(w)\}dv \\ &\leq \int_x^\infty K(u,v)\ \{\Phi(v) - \Phi(w)\}dv \\ &\leq \int_w^\infty K(u,v)\ \{\Phi(v) - \Phi(w)\}dv < \varepsilon.\end{aligned}$$

This completes the proof.

3.4 *PROOF OF THEOREM 3.2*

PROOF

Set

$$\alpha_1(t) = \sup_{0\le v\le t} f(v)$$

$$\alpha_2(t) = \sup_{0\le v\le t} (-f(v))$$

Suppose f(t) is not bounded. Then

(3.4.1) $\limsup_{t\to\infty} |f(t)| = \infty.$

If $f(t) \to \infty$ as $t \to \infty$, then, by Lemma 3.1, $F(u) \to \infty$ as $u \to \infty$, which is a contradiction of (3.2.6). Similarly, if $f(t) \to -\infty$ as $t \to \infty$, then $F(u) \to -\infty$ as $u \to \infty$. By (3.4.1) at least one of α_1 or α_2 tends to infinity. Moreover, either (i) or (ii) below holds.

(i) There exists a sequence $\{t_k\}$ $(k = 1,2,\ldots)$ tending to infinity, such that $\alpha_1(t_k) \ge \alpha_2(t_k)$ $k = 1,2,\ldots$

(ii) There exists a $t_0 > 0$ such that $\alpha_2(t) > \alpha_1(t)$ for all $t > t_0$.

Case (i). We must have $\alpha_1(t) \to \infty$ as $t \to \infty$. Furthermore, the sequence $\{t_k\}$ may be chosen such that

(3.4.2) $f(t_k) = \alpha_1(t_k) \ge \alpha_2(t_k)$ $k = 1,2,\ldots$. For, if v_k satisfies $0 \le v_k \le t_k$ and $f(v_k) = \alpha_1(t_k)$, then, since α_1 and α_2 are increasing, and f is continuous we have

$$\alpha_1(v_k) = f(v_k) = \alpha_1(t_k) \ge \alpha_2(t_k) \ge \alpha_2(v_k).$$

Hence, we may replace t_k by v_k, and the new sequence clearly tends to infinity.

For any $R > 0$, let $x = x(R)$ represent $\inf\{t_k | \alpha_1(t_k) \geq \alpha_2(t_k)$ and $\alpha_1(t_k) > 2R\}$.

Set

$\Delta(R) = \{t | t > x(R)$ and $f(t) \leq f(x)/2\}$.

Now, since $f(t)$ does not tend to infinity, $\Delta(R) \neq \phi$ for large R. Let $y = y(R) = \inf \Delta(R)$. For any $t \in \Delta(R)$, by Lemma 3.2

$$f(t) - f(x) > -M_1 \{\Phi(t) - \Phi(x)\} - M_2 .$$

Hence

$$\begin{aligned} M_1\{\Phi(t) - \Phi(x)\} &> f(x) - f(t) - M_2 \\ &\geq \frac{f(x)}{2} - M_2 \\ &> R - M_2. \end{aligned}$$

Therefore, by the continuity of Φ, $\Phi(y) - \Phi(x) \to \infty$ as $R \to \infty$ and for sufficiently large R, $y > x$.

Let $u \in (x,y)$ satisfy

$$\Phi(u) = \{\Phi(x) + \Phi(y)\}/2.$$

It is immediate that $\Phi(u) - \Phi(x) \to \infty$ and $\Phi(y) - \Phi(u) \to \infty$ as $R \to \infty$. Write

$$\begin{aligned} F(u) &= \left(\int_0^x + \int_x^y + \int_y^\infty\right) K(u,v)\, f(v)dv \\ &= I_1(u) + I_2(u) + I_3(u) \end{aligned}$$

and consider each of I_1, I_2, I_3 in turn.

For $0 \leq v \leq x$, $-f(v) \leq \sup_{0 \leq t \leq x} (-f(t)) = \alpha_2(x)$.

Hence $f(v) \geq -\alpha_2(x)$ for $0 \leq v \leq x$ and

$$(3.4.3) \quad I_1(u) \geq -\alpha_2(x) \int_0^x K(u,v)\,dv$$

$$\geq -\alpha_1(x) \int_0^x K(u,v)\,dv$$

$$= -\alpha_1(x)\ o(1) \text{ as } R \to \infty$$

by (3.2.4).

Consider $I_2(u)$. We have

$$(3.4.4) \quad I_2(u) = \int_x^y K(u,v) f(v)\,dv$$

$$\geq \frac{f(x)}{2} \int_x^y K(u,v)\,dv$$

$$= \frac{\alpha_1(x)}{2} \{1 - o(1) - \int_0^x K(u,v)\,dv - \int_y^\infty K(u,v)\,dv\}$$

$$= \alpha_1(x)\{1/2 - o(1)\} \quad \text{as } R \to \infty$$

by (3.2.4) and Lemma 3.3.

For $I_3(u)$, suppose $v > y > 1$. Then $f(v) - f(y-1) > -M_1\{\Phi(v) - \Phi(y-1)\} - M_2$. But $f(y-1) > \frac{f(x)}{2} > R > M_2 + 1$ for large enough R. Hence, $f(v) > -M_1\ \Phi(v) + M_1\ \Phi(y-1) + 1$. By (3.2.3), for R large enough $\Phi(y) - \Phi(y-1) < 1/M_1$. That is $M_1\ \Phi(y-1) + 1 > M_1\Phi(y)$. Hence $f(v) > -M_1\ \{\Phi(v) - \Phi(y)\}$. Therefore

$$(3.4.5) \quad I_3(u) = \int_y^\infty K(u,v)\ f(v)dv$$

$$> -M_1 \int_y^\infty K(u,v)\ \{\phi(v) - \phi(y)\}dv$$

$$= -o(1) \text{ as } R \to \infty.$$

Combining (3.4.3), (3.4.4), and (3.4.5) we have

$$F(u) > \alpha_1(x)\ \{1/2 - o(1)\} - o(1)$$

$$\to \infty \text{ as } R \to \infty$$

since $\alpha_1(t) \to \infty$ as $t \to \infty$.

This is in contradiction of (3.2.6).

Case (ii). We have $\alpha_2(t) > \alpha_1(t)$ for $t > t_0$. This implies that $\alpha_2(t) \to \infty$ as $t \to \infty$. Take $R > 0$ and let

$y = y(R) = \inf \{t | t > t_0$ and $-\alpha_2(t) = f(t) \leq -2R\}$.

Set

$\Delta(R) = \{t | 0 \leq t \leq y$ and $f(t) > f(y)/2 = -\alpha_2(y)/2\}$.

Note that $\Delta(R)$ is non-empty for large R since $f(y) \to -\infty$ as $R \to \infty$. Let $x = x(R) = \sup \Delta(R)$. Now $x(R) \to \infty$ as $R \to \infty$. Let $t \in \Delta(R)$. Then, by Lemma 3.2

$$M_1\{\Phi(y) - \Phi(t)\} > f(t) - f(y) - M_2$$

$$> -\frac{f(y)}{2} - M_2.$$

Now, $-f(y)/2 \geq R$. Hence,

$$M_1\{\Phi(y) - \Phi(x)\} \geq R - M_2$$

and it follows that $x < y$ and $\Phi(y) - \Phi(x) \to \infty$ as $R \to \infty$.

Let $u \in (x,y)$ satisfy

$$\Phi(u) = \{\Phi(x) + \Phi(y)\}/2.$$

Then $\Phi(y) - \Phi(u) \to \infty$ and $\Phi(u) - \Phi(x) \to \infty$ as $R \to \infty$.

We again write

$$F(u) = \left(\int_0^x + \int_x^y + \int_y^\infty\right) K(u,v)\, f(v)dv$$

$$= I_1(u) + I_2(u) + I_3(u).$$

Then,

$$I_1(u) = \int_0^x K(u,v)\, f(v)dv$$

$$\leq \alpha_1(x) \int_0^x K(u,v)dv$$

$$\leq \alpha_2(x) \int_0^x K(u,v)dv$$

$$\leq \alpha_2(y)\, o(1) \text{ as } R \to \infty.$$

When $x < v < y$, $f(v) \leq -\alpha_2(y)/2$.

Hence

$$I_2(u) \leq -\frac{\alpha_2(y)}{2} \int_x^y K(u,v)dv$$

$$= -\frac{\alpha_2(y)}{2} \{1 - o(1) - \int_0^x K(u,v)dv$$

$$- \int_y^\infty K(u,v)dv\}$$

$$= -\alpha_2(y)\{1/2 - o(1)\} \text{ as } R \to \infty.$$

It remains to consider $I_3(u)$. When $v > t_0$ we have $f(v) \leq \alpha_1(v) \leq \alpha_2(v)$. Hence,

$$I_3(u) \leq \int_y^\infty K(u,v)\ \alpha_2(v)dv$$

$$= \int_y^\infty K(u,v)\ \{\alpha_2(v) - \alpha_2(y) + \alpha_2(y)\}dv.$$

We now show that for $v > y$

(3.4.6) $\alpha_2(v) - \alpha_2(y) < M_1\{\Phi(v) - \Phi(y)\} + M_2.$

First, for $v > y$

$$\begin{aligned} -f(v) - \alpha_2(y) &= -f(v) + f(y) \\ &= -(f(v) - f(y)) \\ &< M_1\{\Phi(v) - \Phi(y)\} + M_2. \end{aligned}$$

Now $\alpha_2(v) = \max\{\alpha_2(y), \sup\limits_{y \leq t \leq v} (-f(t))\}$. If $\alpha_2(v) = \alpha_2(y)$, the left side of (3.4.6) is zero, and we are done. Otherwise, $\alpha_2(v) = -f(t_1)$ where $y < t_1 \leq v$. Then,

$$\begin{aligned} \alpha_2(v) - \alpha_2(y) &< M_1\{\Phi(t_1) - \Phi(y)\} + M_2 \\ &\leq M_1\{\Phi(v) - \Phi(y)\} + M_2 \end{aligned}$$

and (3.4.6) follows.

Hence

$$I_3(u) \leq M_1 \int_y^\infty K(u,v)\{\Phi(v) - \Phi(y)\}dv$$

$$+ M_2 \int_y^\infty K(u,v)dv + \alpha_2(y) \int_y^\infty K(u,v)dv$$

$$= \alpha_2(y)\ o(1) + O(1) \text{ as } R \to \infty.$$

It follows that

$$F(u) \leq -\alpha_2(y)\ \{1/2 - o(1)\} + o(1)$$

$$\rightarrow -\infty \text{ as } R \rightarrow \infty.$$

This is a contradiction of (3.2.6) and the theorem follows.

3.5 *A TAUBERIAN THEOREM OF WIENER*

In this section we state a version of a theorem due to Wiener and found in [7], [18] and [16].

THEOREM 3.3

If

(3.5.1) $g(t)$ *is Lebesgue integrable over the open interval* $(0, \infty)$,

(3.5.2) $\int_0^\infty g(t)t^{-ix}\,dt \neq 0$ *for any real* x,

(3.5.3) $f(t)$ *is slowly decreasing,*

(3.5.4) $f(t)$ *is bounded for* $t \geq 0$, *and*

(3.5.5) $$\lim_{y\to\infty} 1/y \int_0^\infty g(t/y)\ f(t)\ dt = s\int_0^\infty g(t)\ dt$$

then $f(t) \rightarrow s$ *as* $t \rightarrow \infty$.

3.6 *PROOF OF THEOREM 3.1*

By (1.4.2) we have

$$\sigma_\lambda(y) = \frac{\Gamma(\lambda+\varepsilon+1)}{\Gamma(\lambda+1)\Gamma(\varepsilon)}\ \frac{1}{y}\int_0^y (1-t/y)^{\varepsilon-1}(t/y)^\lambda \sigma_{\lambda+\varepsilon}(t)\ dt.$$

It suffices to show that the conditions of

Theorem 3.3 are satisfied with $g(t) = \frac{\Gamma(\lambda+\varepsilon+1)}{\Gamma(\lambda+1)\Gamma(\varepsilon)} t^{\lambda}(1-t)^{\varepsilon-1}$, for $0 < t < 1$, and zero otherwise, and $f(t) = \sigma_{\lambda+\varepsilon}(t)$.

Note that $\int_0^\infty g(t)dt = 1$.

Clearly, (3.5.1) and (3.5.2) hold. Also, (3.5.3) and (3.5.5) hold by hypotheses of Theorem 3.1. It remains to show that $f(t)$ is bounded for $t \geq 0$.

It suffices to show that the conditions and initial hypotheses of Theorem 3.2 are satisfied with

$K(u,v) = \frac{\Gamma(\lambda+\varepsilon+1)}{\Gamma(\lambda+1)\Gamma(\varepsilon)} \frac{1}{u} (\frac{v}{u})^{\lambda} (1-\frac{v}{u})^{\varepsilon-1}$, for $0 < v < u$, and zero otherwise, and $\phi(x) = x/e$, for $0 \leq x \leq e$, and $\phi(x) = \ln x$ for $e < x < \infty$.

Now, $K(u,v) \geq 0$ and $\int_0^\infty K(u,v)dv = 1$ for $u > 0$. Moreover, since $f(v)$ is continuous, $\int_0^\infty K(u,v)f(v)\,dv$ exists for each $u > 0$.

Conditions (3.2.1), (3.2.2), (3.2.3) and (3.2.6) are evidently satisfied. Also, since $K(u,v) = 0$ whenever $v \geq u > 0$, (3.2.5) holds. It remains to show that (3.2.4) holds. For $u > x$ we have

$$\int_0^x K(u,v)dv = \frac{M}{u}\int_0^x (\frac{v}{u})^{\lambda} (1-\frac{v}{u})^{\varepsilon-1} dv$$

$$= M\int_0^{x/u} t^{\lambda} (1-t)^{\varepsilon-1} dt$$

$$\to 0 \text{ as } x/u \to 0$$

and hence as $\ln \frac{u}{x} \to \infty$.

This completes the proof of Theorem 3.1.

3.7 *A TAUBERIAN THEOREM OF O-TYPE*

In this section we prove the following result:

THEOREM 3.4

For $t > 0$ *and* $\mu > \lambda > -1$, *if* $\sum_{n=0}^{\infty} a_n = s(A_\lambda)$ *and* $\sigma_\mu(t) = O(1)$, *then* $\sum_{n=0}^{\infty} a_n = s(A_{\lambda+\delta})$ *for* $0 \leq \delta < \mu - \lambda$.

The proof will be deduced from Theorem 3.1 and two additional lemmas.

LEMMA 3.4

If f *is Lebesgue-integrable on the interval* $(-\infty,\infty)$, *then*

$$\int_{-\infty}^{\infty} |f(rx) - f(x)|dx \to 0 \text{ as } r \to 1.$$

PROOF

Take $\varepsilon > 0$, and suppose $\frac{1}{2} \leq r \leq 2$. For any $M \geq 0$,

$$\int_{-\infty}^{-M} |f(rx) - f(x)|dx + \int_{M}^{\infty} |f(rx) - f(x)|dx$$

$$\leq \frac{1}{r}\int_{-\infty}^{-rM} |f(x)|dx + \int_{-\infty}^{-M} |f(x)|dx$$

$$+ \frac{1}{r}\int_{rM}^{\infty} |f(x)|dx + \int_{M}^{\infty} |f(x)|dx$$

$$\leq 3\int_{-\infty}^{-M/2} |f(x)|dx + 3\int_{M/2}^{\infty} |f(x)|dx$$

Now, since $f(x) \in L(-\infty,\infty)$, there exists an $M_0 > 0$, such that whenever $M > M_0$

$$3\int_{-\infty}^{-M/2} |f(x)|dx + 3\int_{M/2}^{\infty} |f(x)|dx < \varepsilon/2.$$

Furthermore, since $f(x) \in L(-\infty,\infty)$, there exists a function $g(x)$, continuous on the real line, such that

$$\int_{-\infty}^{\infty} |f(x) - g(x)|dx < \varepsilon/12.$$

For a fixed $M > M_0$, since $g(x)$ is uniformly continuous on the interval $[-2M, 2M]$, there exists a $\delta_1 > 0$ such that

$$|g(rx) - g(x)| < \frac{\varepsilon}{12M}$$

whenever $|rx - x| < \delta_1$ and $x \in [-M,M]$ (hence $rx \in [-2M,2M]$).

Let $\delta = \min\{\frac{\delta_1}{M}, \frac{1}{2}\}$. Then, whenever $x \in [-M,M]$ and $|r-1| < \delta$,

$$\int_{-M}^{M} |f(rx) - f(x)|dx$$

$$\leq \int_{-M}^{M} |f(rx) - g(rx)|dx + \int_{-M}^{M} |g(rx) - g(x)|dx$$

$$+ \int_{-M}^{M} |g(x) - f(x)|dx.$$

$$\leq \frac{1}{r}\int_{-rM}^{rM} |f(x) - g(x)|dx + \frac{\varepsilon}{12M}(2M) + \frac{\varepsilon}{12}$$

$$< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}.$$

Therefore, $\int_{-\infty}^{\infty} |f(rx) - f(x)|dx < \varepsilon$ whenever $|r-1| < \delta$.

This completes the proof.

LEMMA 3.5

If

(3.7.1) $h(t) \in L(0,1)$, *and*

(3.7.2) $f(t)$ *is measurable and bounded for* $t \geq 0$, *then the function* $F(y)$, *defined for* $y > 0$, *by*

$F(y) = \frac{1}{y}\int_0^y h(t/y)f(t)dt$ *is slowly decreasing.*

PROOF

It suffices to show for $y > x > 0$, that $|F(y) - F(x)| \to 0$ as $y/x \to 1$.

$$|F(y) - F(x)|$$

$$= \left|\frac{1}{y}\int_0^y h(t/y)f(t)dt - \frac{1}{x}\int_0^x h(t/x)f(t)dt\right|$$

$$= \left|\frac{1}{y}\int_0^x h(t/y)f(t)dt - \frac{1}{x}\int_0^x \{h(t/y)+h(t/x)-h(t/y)\}f(t)dt + \frac{1}{y}\int_x^y h(t/y)f(t)dt\right|$$

$$\leq \frac{M(y-x)}{xy}\int_0^x |h(t/y)|dt + \frac{M}{x}\int_0^x |h(t/y) - h(t/x)|dt + \frac{M}{y}\int_x^y |h(t/y)|dt$$

$$= M\left(\frac{y}{x} - 1\right) \int_0^{x/y} |h(t)|dt + M\int_0^1 \left|h\left(\frac{x}{y} t\right) - h(t)\right|dt$$

$$+ M\int_{x/y}^{1} |h(t)|dt$$

$\to 0$ as $x/y \to 1$ by Lemma 3.4 and (3.7.1).

This completes the proof of the lemma.

PROOF OF THEOREM 3.4.

By Theorem 3.1, it suffices to show that $\sigma_{\lambda+\delta}(t)$ is slowly decreasing. By (1.4.2) we have

$$\sigma_{\lambda+\delta}(y) = \frac{1}{y}\int_0^y h(t/y)\ \sigma_\mu(t)dt$$

where $h(t) = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\lambda-\delta)\Gamma(\lambda+\delta+1)}\ t^{\lambda+\delta}(1-t)^{\mu-\lambda-\delta-1}$

for $0 < t < 1$ and is zero otherwise. Then since $\sigma_\mu(t)$ is bounded for $t \geq 0$ and $h(t) \in L(0,1)$, the result follows by Lemma 3.5.

NOTE. If $\{s_n\}$ is the sequence defined

by $(1-x)^{\mu+1} \sum_{n=0}^{\infty} \epsilon_n^\mu s_n x^n = \sin\frac{1}{1-x}$ for $|x| < 1$, then

$\sigma_\mu(t) = \sin(1+t)$ is bounded for $t > 0$ and $\{s_n\}$ is not (A_μ)-convergent. But for $-1 < \lambda < \mu$, $s_n \to o(A_\lambda)$ since

$$\sigma_\lambda(y) = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\lambda)\Gamma(\lambda+1)}\frac{1}{y}\int_0^y (1-t/y)^{\mu-\lambda-1}(t/y)^\lambda \sin(1+t)dt$$

$$= \frac{\Gamma(\mu+1)}{\Gamma(\mu-\lambda)\Gamma(\lambda+1)}\int_0^1 (1-u)^{\mu-\lambda}\ u^\lambda\ \sin(1+yu)du$$

$\to 0$ as $y \to \infty$ by the Riemann-Lebesgue Theorem. Hence, Theorem 3.4 does not hold for $\delta = \mu-\lambda$.

CHAPTER 4

A TAUBERIAN THEOREM OF SLOWLY DECREASING TYPE – II

4.1 *INTRODUCTION*

In this chapter we consider the following proposition:

PROPOSITION

For $\lambda > -1$, *if* $\sum_{n=0}^{\infty} a_n = s(L)$ *and* $\sigma_\lambda(t)$ *is slowly decreasing, then* $\sum_{n=0}^{\infty} a_n = s(A_\lambda)$.

We consider the case $\lambda = 0$ and show, first, that the proposition, as stated, is false; and second, that with a modified definition of slowly decreasing, the proposition is true.

4.2 *SOME LEMMAS*

LEMMA 4.1

If $\sigma_0(t) = \frac{1}{1+t} \sum_{n=0}^{\infty} s_n \left(\frac{t}{1+t}\right)^n$ *is convergent for all* $t > 0$, *then*

$$L(y) = \frac{1}{\log(1+y)} \int_0^y \frac{\sigma_0(t)}{1+t}\, dt \qquad y > 0.$$

PROOF

We have

$$\int_0^y \frac{\sigma_0(t)}{1+t}\,dt = \int_0^y \frac{1}{(1+t)^2} \sum_{n=0}^{\infty} s_n \left(\frac{t}{1+t}\right)^n dt$$

$$= \sum_{n=0}^{\infty} s_n \int_0^y \left(\frac{t}{1+t}\right)^n \frac{1}{(1+t)^2}\,dt$$

$$= \sum_{n=0}^{\infty} \frac{s_n}{n+1} \left(\frac{y}{1+y}\right)^{n+1}.$$

The convergence of $\sum_{n=1}^{\infty} s_n \left(\frac{t}{1+t}\right)^n$ for $t > 0$ implies its absolute convergence for $t > 0$, which justifies the term by term integration. This completes the proof.

LEMMA 4.2

If $f(x)$ *is absolutely continuous on* $[0,T]$ *for each* $T > 0$ *and* $f'(x) > -M/x$ *for all* $x > 0$, *then* $f(x)$ *is slowly decreasing.*

PROOF

For $\varepsilon > 0$, and $y > x > 0$

$$f(y) - f(x) = \int_x^y f'(t)\,dt$$

$$> -M \int_x^y \frac{dt}{t}$$

$$= -M \ln y/x > -\varepsilon$$

whenever $y/x < \exp[\varepsilon/M]$. This completes the proof.

4.3 *A COUNTEREXAMPLE*

THEOREM 4.1

There exists a sequence $\{s_n\}$ *such that* $\{s_n\}$ *is* (L)*-convergent, and* $\sigma_0(t)$ *is slowly decreasing, but* $\{s_n\}$ *is not* (A_λ)*-convergent for any* $\lambda > -1$.

PROOF

Let $\{s_n\}$ be the sequence such that

$$(1-x)\sum_{n=0}^{\infty} s_n x^n = R\{(1-x)^{-i}\} \qquad |x| < 1.$$

The power series is convergent for $|x| < 1$ and $\sigma_0(t) = R\{(1+t)^i\}$.

Now

$$t\,\sigma_0'(t) = -\sin(\log(1+t))\,\frac{t}{1+t}$$

$$> -1 \qquad \text{for } t > 0.$$

Hence, by Lemma 4.2, $\sigma_0(t)$ is slowly decreasing.

Clearly, $\{s_n\}$ is not (A_0)-convergent. Furthermore, by Theorems 1.2 and 3.1, $\{s_n\}$ cannot be (A_λ)-convergent for any $\lambda > -1$.

However,

$$L(y) = \frac{1}{\log(1+y)}\int_0^y \frac{\cos(\log(1+t))}{1+t}\,dt$$

$$= \frac{\sin(\log(1+y))}{\log(1+y)} \to 0 \text{ as } y \to \infty.$$

Hence, $s_n \to 0\,(L)$.

This completes the proof.

4.4 *A TAUBERIAN THEOREM FOR* (L) *TO* (A_λ) *SUMMABILITY*

For the remainder of this chapter, we suppose that the slow decrease of a real-valued function, $f(t)$, is defined by the following condition:

(4.4.1) given $\varepsilon > 0$, there exist X, $\delta > 0$ such that $f(y) - f(x) > -\varepsilon$ whenever $y \geq x \geq X$ and $\ln\ln y - \ln\ln x < \delta$.

THEOREM 4.2.

If $\alpha > -1$, $s_n \to s(L)$, *and condition* (4.4.1) *holds with* $f(t) = \sigma_\alpha(t)$, *then* $s_n \to s(A_\alpha)$.

4.5 *METHODS OF SUMMABILITY BASED ON POWER SERIES*

Suppose that

$$p_n \geq 0,\ q_n \geq 0,\ \sum_{v=n}^{\infty} p_v > 0,\ \sum_{v=n}^{\infty} q_v > 0 \text{ for } n = 0,1,2,\ldots .$$

Let

$p(x) = \sum_{n=0}^{\infty} p_n x^n$, $q(x) = \sum_{n=0}^{\infty} q_n x^n$ and denote the radii of convergence of these power series by ρ_p and ρ_q respectively.

We also write

$$p_s(x) = \frac{1}{p(x)} \sum_{n=0}^{\infty} p_n s_n x^n$$

$$q_s(x) = \frac{1}{q(x)} \sum_{n=0}^{\infty} q_n s_n x^n.$$

If $\rho_p > 0$, $\sum_{n=0}^{\infty} p_n s_n x^n$ is convergent in the interval $(0, \rho_p)$ and $\lim_{x \to \rho_p -} p_s(x) = s$, then we say that $\{s_n\}$

is summable to s by the *power series method* (P) and write $s_n \to s(P)$. The method (Q) is defined similarly.

We note that if $q_n = \frac{1}{n+1}$ for $n = 0,1,2,\ldots$, then the method (Q) is the logarithmic method (L).

If $\chi(t)$ is a function of bounded variation on [0,1], the associated normalized function $\chi^*(t)$ is defined as follows:

$$\chi^*(t) = \begin{cases} 0 & t = 0 \\ \frac{1}{2}[\chi(t+)+\chi(t-)] - \chi(0) & 0 < t < 1 \\ \chi(1) - \chi(0) & t = 1 \end{cases}$$

A sequence $\{\mu_n\}$ is said to be an *m-sequence* (moment sequence) if

$$\mu_n = \int_{0^-}^{1} t^n \, d\,\chi(t) \qquad (n = 0,1,2,\ldots)$$

where $\chi(t)$ is of bounded variation in [0,1]. If, in addition,

$$\mu_n \geq \delta \int_0^1 t^n |d\,\chi^*(t)|$$

$(0 < \delta \leq 1;\ n = N, N+1, \ldots)$ then we call $\{\mu_n\}$ an $\overline{m}$*-sequence*.

We note that if $\chi(t)$ is of bounded variation in [0,1], then ([15], Theorem 8a)

$$\int_0^1 t^n \, d\,\chi(t) = \int_0^1 t^n \, d\,\chi^*(t)$$

and ([16] p. 14 and [8] p. 335)

$$\int_0^1 t^n |d\,\chi(t)| \geq \int_0^1 t^n \, |d\,\chi^*(t)|.$$

Hence an m-sequence $\{\mu_n\}$ such that

$$\mu_n = \int_0^1 t^n \, d\,\chi(t) \geq \delta \int_0^1 t^n |d\,\chi(t)|$$

where $0 < \delta \leq 1$ and $n = N, N+1, \ldots$ is necessarily an $\bar{m}$-sequence.

LEMMA 4.3

An m-sequence $\{\mu_n\}$ which converges to a positive limit is an $\bar{m}$-sequence.

PROOF

Suppose, as we may, that

$$\mu_n = \int_0^1 t^n \, d\,\chi(t) = \int_0^1 t^n \, d\alpha(t) - \int_0^1 t^n \, d\beta(t)$$

$$\to u - v > 0 \text{ as } n \to \infty.$$

where $\alpha(t)$, $\beta(t)$ are non-decreasing, and bounded in $[0,1]$ and $u > v \geq 0$.

Then

$$\frac{\mu_n}{\int_0^1 t^n |d\,\chi(t)|} = \frac{\int_0^1 t^n \, d\alpha(t) - \int_0^1 t^n \, d\beta(t)}{\int_0^1 t^n \, d\alpha(t) + \int_0^1 t^n \, d\beta(t)}$$

$$\to \frac{u - v}{u + v}$$

$$\geq \frac{1}{2}\,\frac{u - v}{u + v} = \delta$$

for sufficiently large n. The lemma follows by the observation immediately preceding the lemma.

We state two additional results required for proofs in the next section. Both are due to Borwein.

LEMMA 4.4

If $\infty > \rho_p > 0$, *then a necessary and sufficient condition for* (P) *to be regular is that*

$$\sum_{n=0}^{\infty} p_n (\rho_p)^n = \infty$$

(See [3], Theorem 1)

LEMMA 4.5

If $p_n = \mu_n q_n$ $(n \geq N)$, *where* $\{\mu_n\}$ *is an* $\overline{m}$*-sequence, if* $\rho_p = \rho_q > 0$ *and* (P) *is regular, then* (Q) $\subseteq$ (P).

(See [3], Theorem A').

4.6 *SOME PRELIMINARY LEMMAS*

For the remainder of this chapter we set, for $\alpha > -1$ and $w > 0$,

$$(4.6.1) \quad J_\alpha(w) = \frac{1}{\log(1+w)} \int_0^w (1+t)^{\alpha-1} \left[\log \frac{w(1+t)}{t(1+w)}\right]^\alpha \sigma_\alpha(t)\,dt$$

and

$$(4.6.2) \quad v_n = \frac{\varepsilon_n^\alpha \, \Gamma(\alpha+1)}{(n+1)^\alpha}$$

LEMMA 4.7

For any sequence $\{s_n\}$ *we have* $s_n \to s(L)$ *if and only if* $v_n s_n \to s(L)$.

PROOF.

Suppose $s_n \to s(L)$. Set $q_n = \frac{1}{n+1}$ and $p_n = v_n q_n$ for $n = 0,1,2,\ldots$.

Now since $v_n \to 1$, by Lemma 4.2 (and [7], Theorem 211) $\{v_n\}$ is an $\overline{m}$-sequence. Also, since $p_n \sim \frac{1}{n+1}$ as $n \to \infty$, we have by Lemma 4.4 that the method (P) is regular, since $\sum_{n=0}^{\infty} \frac{1}{n+1} = \infty$. Furthermore, $\rho_p = \rho_q = 1$. Hence, by Lemma 4.5 $(Q) \subseteq (P)$.

Since (Q) is (L)

$$\frac{1}{p(x)} \sum_{n=0}^{\infty} s_n \frac{v_n}{n+1} x^n \to s \text{ as } x \to 1^-.$$

Since $v_n \to 1$ and (L) is regular

$$\frac{-xp(x)}{\log(1-x)} = -\frac{1}{\log(1-x)} \sum_{n=0}^{\infty} \frac{v_n}{n+1} x^{n+1} \to 1 \text{ as } x \to 1^-.$$

Therefore

$$-\frac{1}{\log(1-x)} \sum_{n=0}^{\infty} s_n \frac{v_n}{n+1} x^{n+1} = \left\{\frac{-xp(x)}{\log(1-x)}\right\}\left\{\frac{1}{p(x)} \sum_{n=0}^{\infty} s_n \frac{v_n}{n+1} x^n\right\}$$

$$\to s \text{ as } x \to 1^-.$$

That is

$$v_n s_n \to s(L).$$

Now suppose $v_n s_n \to s(L)$. Taking $h_n = 1/v_n$ and considering the power series methods (Q) and (P) where $q_n = \frac{1}{n+1}$ and $p_n = h_n q_n$, we can show that $s_n \to s(L)$ by an argument similar to that above.

This completes the proof.

LEMMA 4.8

If $\sigma_\alpha(t)$ *is convergent for all* $t > 0$, *then* $J_\alpha(w) \to s$ *as* $w \to \infty$, *if and only if* $v_n s_n \to s(L)$.

PROOF

Setting $u = \frac{t(1+w)}{w(1+t)}$ in $J_\alpha(w)$ gives $J_\alpha(w)$

$$= \frac{1}{\log(1+w)} \int_0^w (1+t)^{\alpha-1}(1+t)^{-\alpha-1} \sum_{n=0}^{\infty} \varepsilon_n^\alpha s_n \left(\frac{t}{1+t}\right)^n \left[\log \frac{w(1+t)}{t(1+w)}\right]^\alpha dt$$

$$= \frac{1}{\log(1+w)} \int_0^1 \left[\sum_{n=0}^{\infty} \varepsilon_n^\alpha s_n \left(\frac{w}{1+w}\right)^{n+1} u^n\right] \left[\log \frac{1}{u}\right]^\alpha du$$

$$= \frac{1}{\log(1+w)} \sum_{n=0}^{\infty} \varepsilon_n^\alpha s_n \left(\frac{w}{1+w}\right)^{n+1} \int_0^1 u^n \left(\log \frac{1}{u}\right)^\alpha du$$

$$= \frac{\Gamma(\alpha+1)}{\log(1+w)} \sum_{n=0}^{\infty} \frac{\varepsilon_n^\alpha}{(n+1)^{\alpha+1}} s_n \left(\frac{w}{1+w}\right)^{n+1}$$

$$= \frac{1}{\log(1+w)} \sum_{n=0}^{\infty} \frac{v_n s_n}{n+1} \left(\frac{w}{1+w}\right)^{n+1}.$$

Now, the convergence of the series defining $\sigma_\alpha(t)$ for $t > 0$ implies its absolute convergence. This justifies the integration term by term and the proof is complete.

LEMMA 4.9

If $\alpha > -1$, $\sigma_\alpha(t)$ *satisfies condition* (4.4.1) *and* $s_n \to s(L)$ *then* $\sigma_\alpha(t) = O(1)$ $(t > 0)$.

PROOF

Since it suffices to show that the result holds for $s = 0$, we have by Lemma 4.7 $v_n s_n \to 0(L)$, and hence by Lemma 4.8 $J_\alpha(w) \to 0$ as $w \to \infty$.

We show that the pre-conditions and conditions of Theorem 3.2 are satisfied with

$$K(w,t) = \begin{cases} \frac{1}{\log(1+w)} (1+t)^{\alpha-1} [\log \frac{w(1+t)}{t(1+w)}]^{\alpha} & 0<t<w \\ 0, & \text{otherwise} \end{cases}$$

$$\Phi(x) = \begin{cases} x/e^{e} & 0 \le x < e^{e} \\ \ln\ln x & e^{e} \le x \end{cases}$$

and

$$f(t) = \sigma_{\alpha}(t).$$

Since

$$\int_0^\infty K(w,t)dt = \frac{1}{\log(1+w)} \int_0^w (1+t)^{\alpha-1} [\log \frac{w(1+t)}{t(1+w)}]^{\alpha} dt$$

$$= \frac{1}{\log(1+w)} \int_0^w (1+t)^{\alpha-1} [\log \frac{w(1+t)}{t(1+w)}]^{\alpha} (1+t)^{-\alpha-1} \sum_{n=0}^{\infty} \varepsilon_n^{\alpha} (\frac{t}{1+t})^n dt$$

we have, as in the proof of Lemma 4.8

$$\int_0^\infty K(w,t)\, dt = \frac{1}{\log(1+w)} \sum_{n=0}^{\infty} \frac{v_n}{n+1} (\frac{w}{1+w})^{n+1}$$

$$\to 1 \text{ as } w \to \infty$$

since $v_n \to 1$ as $n \to \infty$ and the (L) method is regular. This establishes the pre-conditions of Theorem 3.2.

Conditions (3.2.1), (3.2.2), and (3.2.3) clearly hold.

Furthermore, since $K(w,t) = 0$ whenever $t \ge w$, (3.2.5) holds. It remains to show that (3.2.4) is satisfied, since (3.2.6) holds by the hypothesis, $s_n \to 0(L)$.

For $\alpha \geq 0$ and $x \geq 1$, we have

$$\log\ (1+w) \int_0^x K(w,t)\ dt$$

$$= \int_0^x (1+t)^{\alpha-1} [\log \frac{w(1+t)}{t(1+w)}]^{\alpha} dt$$

$$\leq \int_0^x (1+t)^{\alpha-1} [\log \frac{1+t}{t}]^{\alpha} dt$$

$$= \left\{ \int_0^1 + \int_1^x \right\} (1+t)^{\alpha-1} (\log \frac{1+t}{t})^{\alpha} dt$$

$$= I_1 + I_2.$$

Setting $u = 1/t$ in I_1 gives

$$I_1 = \int_1^{\infty} (1+1/u)^{\alpha-1} [\log (1+u)]^{\alpha} \frac{du}{u^2}$$

$$= O(1) = o(\log\ (1+w))$$

$$I_2 = \int_1^x (1+t)^{\alpha-1} [\log\ (1+1/t)]^{\alpha}\ dt$$

$$= O(1) \int_1^x (1+t)^{-1}\ dt$$

$$= O(1) \log\ (1+x) + O(1).$$

Therefore $\frac{1}{\log(1+w)} \{I_1 + I_2\}$

$$= o(1) + O(1) \frac{\log(1+x)}{\log(1+w)}$$

$$= o(1) \text{ as } w > x \to \infty \text{ and } \ln\ln w - \ln\ln x \to \infty.$$

Hence, condition (3.2.4) holds for $\alpha \geq 0$.

In the case $-1 < \alpha < 0$, we first observe that for $0 < t < w$ we have

$$\log \frac{w(1+t)}{t(1+w)} > \frac{w-t}{w(1+t)}$$

since, for $y > 1$

$$\log y = \log y - \log 1 = \frac{y-1}{\theta} \qquad (1 < \theta < y)$$

$$> \frac{y-1}{y}$$

and for $0 < t < w$, $\frac{w(1+t)}{t(1+w)} > 1$.

Hence

$$\frac{1}{\log(1+w)} \int_0^x (1+t)^{\alpha-1} [\log \frac{w(1+t)}{t(1+w)}]^{\alpha} dt$$

$$\leq \frac{1}{\log(1+w)} \int_0^x (1+t)^{\alpha-1} [\frac{w-t}{w(1+t)}]^{\alpha} dt$$

$$= \frac{1}{\log(1+w)} \int_0^x (1-t/w)^{\alpha} \frac{dt}{1+t}$$

$$\leq \frac{(1-x/w)^{\alpha}}{\log(1+w)} \int_0^x \frac{dt}{1+t}$$

$$= O(1) \frac{\log(1+x)}{\log(1+w)} = o(1) \text{ as}$$

$w > x \to \infty$ and $\ln\ln w - \ln\ln x \to \infty$, since the latter implies $\frac{\ln x}{\ln w} \to 0$ and $x/w \to 0$. This establishes condition (3.2.4) for $\alpha > -1$ and hence the lemma.

We now give a direct proof of Theorem 1.4.

THEOREM 4.3

$$(A_\alpha) \subseteq (L) \textit{ for } \alpha > -1$$

PROOF

The theorem is an immediate consequence of Lemmas 4.7, 4.8 and 3.1 with

$$K(w,t) = \begin{cases} \dfrac{1}{\log(1+w)}(1+t)^{\alpha-1}\left[\log\dfrac{w(1+t)}{t(1+w)}\right]^\alpha & 0 < t < w \\ 0 & \text{otherwise} \end{cases}$$

$$f(t) = \sigma_\alpha(t) \qquad t > 0$$

and

$$\phi(x) = \begin{cases} x/e^e & 0 \le x < e^e \\ \ln\ln x & e^e \le x \end{cases}$$

since condition (3.2.4) is established as in Lemma 4.9 and the other conditions are clearly satisfied.

LEMMA 4.10

If $\gamma > 1$, *and* $\alpha > -1$ *then*

$$I(x) = \int_0^x (1+t)^{\alpha-1}\left\{\left[\log\frac{x^\gamma(1+t)}{t(1+x^\gamma)}\right]^\alpha - \left[\log\frac{x(1+t)}{t(1+x)}\right]^\alpha\right\}dt = O(1).$$

PROOF

Suppose $\alpha \ge 1$. Then if $x \ge 1$

$$|I(x)| = I(x) \le \alpha \log\frac{x^\gamma(1+x)}{x(1+x^\gamma)}\int_0^x (1+t)^{\alpha-1}\left[\log\frac{x^\gamma(1+t)}{t(1+x^\gamma)}\right]^{\alpha-1}dt$$

$$\le \alpha \log\frac{x^\gamma(1+x)}{(1+x^\gamma)x}\left(\int_0^1 + \int_1^x\right)(1+t)^{\alpha-1}\left(\log\frac{1+t}{t}\right)^{\alpha-1}dt.$$

Now,

$$\int_0^1 (1+t)^{\alpha-1}\left(\log\frac{1+t}{t}\right)^{\alpha-1} dt = \int_1^\infty (1+1/u)^{\alpha-1}[\log(1+u)]^{\alpha-1}\frac{du}{u^2}$$

$$< \infty.$$

Hence,

$$\alpha \log \frac{x^\gamma(1+x)}{x(1+x^\gamma)} \int_0^1 (1+t)^{\alpha-1} \left(\log\frac{1+t}{t}\right)^{\alpha-1} dt = O(1).$$

Also,

$$\alpha \log \frac{x^\gamma(1+x)}{x(1+x^\gamma)} \int_1^x (1+t)^{\alpha-1} \left(\log \frac{1+t}{t}\right)^{\alpha-1} dt$$

$$= O(1) \log \frac{x^\gamma(1+x)}{x(1+x^\gamma)} \cdot \int_1^x dt$$

$$\leq O(1)\ x \log \frac{1+x}{x} = O(1).$$

Suppose $0 < \alpha < 1$. As established in the proof of Lemma 4.9

$$\log \frac{x(1+t)}{t(1+x)} > \frac{x-t}{x(1+t)} \qquad (0 < t < x).$$

We have

$$|I(x)| = I(x) \leq \alpha \log\frac{x^\gamma(1+x)}{x(1+x^\gamma)} \int_0^x (1+t)^{\alpha-1}\left[\log\frac{x(1+t)}{t(1+x)}\right]^{\alpha-1} dt$$

$$\leq \alpha \frac{M}{x} \int_0^x (1+t)^{\alpha-1}\left[\frac{x-t}{x(1+t)}\right]^{\alpha-1} dt$$

since $\left|x \log \frac{x^\gamma(1+x)}{x(1+x^\gamma)}\right| \leq M$. Therefore

$$|I(x)| \leq \frac{\alpha M}{x^\alpha} \int_0^x (x-t)^{\alpha-1} dt = M.$$

Suppose $-1 < \alpha < 0$. Then

$$|I(x)| = -I(x) = \int_0^{x/2} (1+t)^{\alpha-1}\left\{\left[\log\frac{x(1+t)}{t(1+x)}\right]^{\alpha} - \left[\log\frac{x^{\gamma}(1+t)}{t(1+x^{\gamma})}\right]^{\alpha}\right\}dt$$

$$+ \int_{x/2}^{x} (1+t)^{\alpha-1}\left\{\left[\log\frac{x(1+t)}{t(1+x)}\right]^{\alpha} - \left[\log\frac{x^{\gamma}(1+t)}{t(1+x^{\gamma})}\right]^{\alpha}\right\} dt$$

$$= I_1(x) + I_2(x).$$

As in the case $0 < \alpha < 1$,

$$0 \le I_1(x) \le \alpha \log\frac{x(1+x^{\gamma})}{x^{\gamma}(1+x)} \int_0^{x/2} (1+t)^{\alpha-1} \left[\log\frac{x(1+t)}{t(1+x)}\right]^{\alpha-1} dt$$

$$\le \frac{-\alpha M}{x} \int_0^{x/2} (1+t)^{\alpha-1} \left[\frac{x-t}{x(1+t)}\right]^{\alpha-1} dt$$

since $\left|x \log \left(\frac{x}{1+x} \cdot \frac{1+x^{\gamma}}{x^{\gamma}}\right)\right| \le M$. Therefore

$$I_1(x) \le -\frac{\alpha M}{x^{\alpha}} \int_0^{x/2} (x-t)^{\alpha-1} dt$$

$$= M[(\tfrac{1}{2})^{\alpha} - 1].$$

Finally,

$$0 \le I_2(x) \le \int_{x/2}^{x} (1+t)^{\alpha-1}\left[\log\frac{x(1+t)}{t(1+x)}\right]^{\alpha}dt$$

$$\le \int_{x/2}^{x} (1+t)^{\alpha-1} \left[\frac{x-t}{x(1+t)}\right]^{\alpha} dt$$

$$= \frac{1}{x^{\alpha}} \int_{x/2}^{x} (x-t)^{\alpha} \frac{dt}{1+t}.$$

Now, since $1+t > x/2$

$$I_2(x) \leq \frac{2}{x^{\alpha+1}} \int_{x/2}^{x} (x-t)^{\alpha}\, dt$$

$$= \frac{2}{x^{\alpha+1}} \frac{1}{\alpha+1} \left(\frac{x}{2}\right)^{\alpha+1} = \frac{1}{(\alpha+1)2^{\alpha}}$$

Hence, $I(x) = O(1)$ in this case.

Finally, since the case $\alpha = 0$ is trivial the lemma is established.

LEMMA 4.11

If $\gamma > 1$, and $\alpha > -1$, then

$$\int_{x}^{x^{\gamma}} (1+t)^{\alpha-1} \left[\log \frac{x^{\gamma}(1+t)}{t(1+x^{\gamma})}\right]^{\alpha} dt$$

$$= (\gamma-1)\log(1+x) + o(\log(1+x)).$$

PROOF

If $\sigma_{\alpha}(t) = 1$ for $t > 0$, then by Lemmas 4.7, 4.8 and Theorem 1.4 $J_{\alpha}(x) = 1 + o(1)$ as $x \to \infty$.

Now, by Lemma 4.10

$$\int_{x}^{x^{\gamma}} (1+t)^{\alpha-1} \log\left[\frac{x^{\gamma}(1+t)}{t(1+x^{\gamma})}\right]^{\alpha} dt$$

$$= \left(\int_0^{x^{\gamma}} - \int_0^{x}\right) (1+t)^{\alpha-1} \left[\log \frac{x^{\gamma}(1+t)}{t(1+x^{\gamma})}\right]^{\alpha} dt$$

$$= \log(1+x^{\gamma}) + o(\log(1+x^{\gamma})) - \log(1+x) + o(\log(1+x)) + O(1)$$

$$= (\gamma-1)\log(1+x) + o(\log(1+x)).$$

This establishes the lemma.

4.7 *PROOF OF THEOREM 4.2*

By (4.4.1), for $\varepsilon > 0$, there exist an X, $\delta > 0$ such that

(4.7.1) $\sigma_\alpha(y) - \sigma_\alpha(x) > -\varepsilon$ whenever $y > x > X$ and $\ln\ln y - \ln\ln x < \delta$. That is, writing $\delta = \ln \gamma$, $\sigma_\alpha(x) - \varepsilon < \sigma_\alpha(y)$ whenever $X < x < y < x^\gamma$.

We may, without loss of generality, suppose that $s_n \to 0(L)$. Hence, by Lemma 4.7, $v_n s_n \to 0(L)$. That is, $J_\alpha(x) \to 0$ as $x \to \infty$. Therefore if $x > X$

(4.7.2)
$$\int_x^{x^\gamma} (1+t)^{\alpha-1}\left[\log \frac{x^\gamma(1+t)}{t(1+x^\gamma)}\right]^\alpha [\sigma_\alpha(x) - \varepsilon]\, dt$$

$$\leq \int_x^{x^\gamma} (1+t)^{\alpha-1}\left[\log \frac{x^\gamma(1+t)}{t(1+x^\gamma)}\right]^\alpha \sigma_\alpha(t)\, dt$$

$$= \left[\int_0^{x^\gamma} - \int_0^{x}\right] (1+t)^{\alpha-1} \left[\log \frac{x^\gamma(1+t)}{t(1+x^\gamma)}\right]^\alpha \sigma_\alpha(t)\, dt$$

$$= \log(1+x^\gamma)\, J_\alpha(x^\gamma) - \log(1+x)\, J_\alpha(x) + 0(1)$$

by Lemma 4.10. Now, $J_\alpha(x) \to 0$.

Therefore

(4.7.3)
$$\int_x^{x^\gamma} (1+t)^{\alpha-1}\left[\log \frac{x^\gamma(1+t)}{t(1+x^\gamma)}\right]^\alpha \sigma_\alpha(t)\, dt$$

$$= o(\log(1+x^\gamma)) + o(\log(1+x))$$

$$= o(\log(1+x)).$$

But, by Lemma 4.11,

$$(\sigma_\alpha(x) - \varepsilon) \int_x^{x^\gamma} (1+t)^{\alpha-1} \left[\log \frac{x^\gamma(1+t)}{t(1+x^\gamma)}\right]^\alpha dt$$

$$= (\sigma_\alpha(x) - \varepsilon)[(\gamma-1) \log(1+x) + o(\log(1+x))].$$

Therefore

$$(\sigma_\alpha(x) - \varepsilon)[(\gamma-1) \log(1+x) + o(\log(1+x))] \leq o(\log(1+x)).$$

That is,

$$\sigma_\alpha(x) - \varepsilon \leq \frac{o(1)}{(\gamma-1)+o(1)},$$

and hence

$$(4.7.4) \qquad \limsup_{x\to\infty} \sigma_\alpha(x) \leq \varepsilon.$$

Rewriting (4.7.1) we have

$\sigma_\alpha(x) < \sigma_\alpha(y) + \varepsilon$ whenever $X < y^{1/\gamma} < x < y$.

Furthermore, by (4.7.3)

(replacing x by $x^{1/\gamma}$) we have

$$o(\log(1+y)) = \int_{y^{1/\gamma}}^{y} (1+t)^{\alpha-1} \left[\log \frac{y(1+t)}{t(1+y)}\right]^\alpha \sigma_\alpha(t)\, dt$$

$$\leq \int_{y^{1/\gamma}}^{y} (1+t)^{\alpha-1} \left[\log \frac{y(1+t)}{t(1+y)}\right]^\alpha (\sigma_\alpha(y) + \varepsilon)\, dt$$

$$= (\sigma_\alpha(y) + \varepsilon) \{(1-1/\gamma) \log(1+y) + o(\log(1+y))\}$$

Hence,

$$\frac{o(1)}{(1-1/\gamma)+o(1)} \leq \sigma_\alpha(y) + \varepsilon$$

and therefore

$$(4.7.5) \qquad -\varepsilon \leq \liminf_{y\to\infty} \sigma_\alpha(y)$$

Combining (4.7.4) and (4.7.5) completes the proof of the theorem.

4.8 *A CONDITION ON* $\{na_n\}$ *WHICH IMPLIES RELATION* (4.4.1)

We say $\{s_n\} = O_L(f(y))(A_\lambda)$ if there exist a y_0 and an $M > 0$ such that $f(y)$ is defined and positive for $y \geq y_0$ and $\sigma_\lambda(y) > -M\, f(y)$ for $y \geq y_0$.

We also set

$$t_\lambda(y) = (1+y)^{-\lambda-1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda\, n\, a_n \left(\frac{y}{1+y}\right)^n$$

for $y > 0$.

It is easily shown that

$$y \frac{d}{dy} \sigma_\lambda(y) = (\lambda+1)\{\sigma_{\lambda+1}(y) - \sigma_\lambda(y)\} = t_\lambda(y).$$

Furthermore, as in the proof of Lemma 4.2, $\sigma_\lambda(y)$ satisfies condition (4.4.1) if $\sigma_\lambda'(y) > -M/y\ln y$ for sufficiently large y.

Hence, $\sigma_\lambda(y)$ satisfies (4.4.1) if $\{n\, a_n\} = O_L(1/\ln y)(A_\lambda)$.

CHAPTER 5

A TAUBERIAN THEOREM FOR ABELIAN SUMMABILITY METHODS

5.1 *INTRODUCTION*

In this chapter we assume, as in section 1.5, that

$$0 \le \lambda_1 < \lambda_2 < \cdots; \quad \lambda_n \to \infty$$

and prove a general tauberian theorem for the abelian summability method (A, λ).

THEOREM 5.1

If $\lambda_{n+1}/\lambda_n \to 1$, *if* $\sum_{n=1}^{\infty} a_n$ *is summable* (A, λ) *to* s, *and if* $\liminf \{s_n - s_m\} \ge 0$ *as* $n \ge m \to \infty$ *and* $\lambda_n/\lambda_m \to 1$, *then* $s_n \to s$.

When $\lambda_n = n$, then (A, λ) method is equivalent to the ordinary Abel summability method A_0, and the result in this case is well-known (see, for example, [7], Theorem 106). The only other case which appears to be known is $\lambda_n = \log n$. The result in this case was established by Kwee [10] by a method fundamentally different from and more complicated than ours.

5.2 *A PRELIMINARY LEMMA*

We use the following known result (see [7], p. 164, Theorem 105):

LEMMA 5.1

If α *is a function of bounded variation on every interval* $[0,T]$, *if* $\int_0^\infty e^{-yt}\,d\alpha(t)$ *is convergent for* $y > 0$ *and tends to* s *as* $y \to 0+$, *and if* $\liminf\,\{\alpha(y) - \alpha(x)\} \geq 0$ *as* $y \geq x \to \infty$ *and* $y/x \to 1$, *then* $\alpha(t) \to s$ *as* $t \to \infty$.

5.3 *PROOF OF THEOREM 5.1*

Set

$$\alpha(t) = \sum_{\lambda_n < t} a_n .$$

Then for $y > 0$, we have

$$\int_0^\infty e^{-yt}\,d\alpha(t) = \sum_{n=1}^\infty a_n e^{-y\lambda_n}$$

and, by hypothesis, the series converges and its sum tends to s as $y \to 0+$.

Assign $\varepsilon > 0$. Then there exist positive numbers M, δ such that $s_n - s_m > -\varepsilon$ whenever $n \geq m \geq M$ and $\lambda_n/\lambda_m \leq 1+2\delta$. Choose an integer N such that $\lambda_N > \lambda_{M+1}$ and, for $m+1 \geq N$,

$$\frac{\lambda_{m+1}}{\lambda_m} \leq \frac{1+2\delta}{1+\delta} .$$

Let $y \geq x \geq \lambda_N$ and $y/x \leq 1+\delta$.

Then there exist integers n, m such that $\lambda_{n+1} \geq y > \lambda_n$ and $\lambda_{m+1} \geq x > \lambda_m$. Hence $n \geq m \geq M$ and

$$\frac{\lambda_n}{\lambda_m} < \frac{y}{x} \cdot \frac{\lambda_{m+1}}{\lambda_m} \leq (1+\delta)\,\frac{1+2\delta}{1+\delta} = 1+2\delta ;$$

and therefore

$$\alpha(y) - \alpha(x) = \alpha(\lambda_{n+1}) - \alpha(\lambda_{m+1})$$

$$= s_n - s_m > -\varepsilon.$$

Consequently $\liminf \{\alpha(y) - \alpha(x)\} \geq 0$ as $y \geq x \to \infty$ and $y/x \to 1$; and so, by Lemma 5.1, $s_n \to s$. This completes the proof.

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